Example1 : Find the flux

of

$\vec{F}(x,y,z) = \langle x, 2y, z^2 \rangle$

where S is the

top part of the sphere

with center at the origin

and radius 2.

S is given by the graph of the function

$$f(x,y) = z = \sqrt{4-x^2-y^2},$$

on $R = \{(x,y) \mid x^2+y^2 \leq 4\}$.

We then use the formula, derived in class, that

$$\iint_S \vec{F} \cdot d\vec{S}$$

$$= \iint_R \vec{F} \cdot \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle dA$$

Then

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{4-x^2-y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{4-x^2-y^2}}$$, so we obtain

$$\iint_S \vec{F} \cdot d\vec{S} =$$

$$\iint_R \left(\frac{-x^2}{4-x^2-y^2} - \frac{2y^2}{4-x^2-y^2} + z^2 \right) dA$$

$$= \iint_R \frac{-x^2-2y^2}{\sqrt{4-x^2-y^2}} + (4-x^2-y^2)\, dA$$

Now R is best described using polar coordinates,

$$R = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

The integral then becomes

$(x = r\cos\theta,\ y = r\sin\theta,\ \text{Jacobian} = r)$

$$\int_0^{2\pi}\int_0^2 r\left(\frac{-r^2(\cos^2\theta + 2\sin^2\theta)}{\sqrt{4-r^2}} + 4 - r^2\right) dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(\frac{-r^3(1+\sin^2\theta)}{\sqrt{4-r^2}} + 4 - r^2\right) dr\, d\theta$$

The second integral is

$$\int_0^{2\pi}\int_0^2 (4r - r^3)\, dr\, d\theta$$

$$= 2\pi \left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2$$

$$= 8\pi$$

The first integral is

$$\int_0^{2\pi}\int_0^2 \frac{-r^3(1+\sin^2\theta)}{\sqrt{4-r^2}}\, dr\, d\theta$$

$$= \int_0^2 \int_0^{2\pi} \frac{-r^3\left(\frac{3}{2} - \frac{\cos(2\theta)}{2}\right)}{\sqrt{4-r^2}} \, d\theta \, dr$$

$$= \int_0^2 \left(\frac{-r^3}{\underbrace{\sqrt{4-r^2}}}\right) dr \int_0^{2\pi} \left(\frac{3}{2} - \frac{\cos(2\theta)}{2}\right) d\theta$$

$u = 4 - r^2 \Leftrightarrow r^2 = 4 - u$

$du = -2r\,dr$

$$= \frac{1}{2} \int_4^0 \frac{4-u}{\sqrt{u}} \, du \left(\frac{3\theta}{2} - \frac{\sin(2\theta)}{4}\right)\Big|_0^{2\pi}$$

$$= -\frac{1}{2} \int_0^4 \left(4u^{-1/2} - u^{1/2}\right) du \cdot 3\pi$$

$$= -\frac{3\pi}{2}\left(8u^{1/2} - \frac{2}{3}u^{3/2}\right)\Big|_0^4$$

$$= -\frac{3\pi}{2}\left(\frac{32}{3}\right)$$

$$= -16\pi$$

So the final answer is

$$-16\pi + 8\pi = \boxed{-8\pi}$$

How could we make this integral easier?

Sometimes a surface is bounded by a curve, sometimes it isn't.

When it is, we have:

Stokes' Theorem Let S be an oriented surface with boundary curve C, oriented counterclockwise. Let

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$

be a vector field such that P, Q, and R have continuous first order partials in an open region containing S. Then

$$\int_C \vec{F}(x,y,z)\cdot d\vec{r}$$

$$= \int_S \mathrm{curl}(\vec{F})\cdot d\vec{S}$$

$$= \int_S (\nabla \times \vec{F})\cdot d\vec{S}$$

So this theorem relates
the integral of a
vector field over a
curve to the integral
of its curl over
a surface bounded
by the curve.

Example 2: Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}(x,y,z) = \langle x^2 z, xy^2, z^2 \rangle$

and C is the curve

of intersection of

$x+y+z=1$ and

$x^2+y^2=9$

$$\text{Curl}(\vec{F}) = \langle 0, x^2, y^2 \rangle$$

Now S is given by the graph of a function

$f(x,y) = 1 - x - y$ over the region $R = \{(x,y) \mid x^2 + y^2 \leq 3\}$.

We can use the formula, derived in class, that if $\vec{G} = \langle P, Q, R \rangle$ is any vector field with continuous partials, then

$$\iint_S \vec{G} \cdot d\vec{S}$$

$$= \iint_R \vec{G} \cdot \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle dA$$

with $\vec{G} = \text{curl}(\vec{F}) = \langle 0, x^2, y^2 \rangle$,

we get

$$\iint_R \langle 0, x^2, y^2 \rangle \cdot \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle dA$$

$$= \iint_R \left(-\frac{\partial f}{\partial y} x^2 + y^2 \right) dA$$

$$= \iint_R x^2 + y^2 \, dA$$

Now R is best described using polar coordinates:

$$R = \{(r,\theta) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq 3\}$$

The integral then becomes

($x = r\cos\theta$, $y = r\sin\theta$, Jacobian $= r$)

$$\int_0^3 \int_0^{2\pi} r^3 \, dr \, d\theta$$

$$= 2\pi \int_0^3 r^3 \, dr = \boxed{\frac{81\pi}{2}}$$

The Divergence Theorem

Let E be a solid region in $\mathbb{R}^3$ with boundary surface S, oriented positively (normal vector $\vec{n}$ points out of S). Let $\vec{F} = \langle P, Q, R \rangle$

such that the first order partials of $P, Q,$ and R are continuous in an open region containing E

Then

$$\iint_S \vec{F} \cdot d\vec{S}$$

$$= \iiint_E \operatorname{div}(\vec{F})\, dV$$

$$= \iiint_E \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$$

Physical Interpretations

1) Curl: Let

$\vec{v} = \langle P, Q, R \rangle$ be the velocity vector field of a fluid. $\text{Curl}(\vec{v}) = \langle 0,0,0 \rangle$ at a point means there is no rotation of the fluid about that point. $\text{Curl}(\vec{v}) \neq \langle 0,0,0 \rangle$ means there IS rotation (an object would <u>curl</u> about the point

2) Divergence: Given a fluid, let $\vec{v} = \langle P, Q, R \rangle$ denote its velocity field and suppose the fluid has constant density ρ.

Let $\vec{F} = \rho \cdot \vec{v}$. Then $\vec{F}$ represents the rate of fluid flow per unit area. The divergence of $\vec{F}$ tells whether the fluid flows away ($\mathrm{div}(\vec{F}) > 0$) or to ($\mathrm{div}(\vec{F}) < 0$) a point

3) $\text{div}(\text{curl}(\vec{F})) = 0$

(intuitive, and most likely not quite correct)

The curl represents "rotation" which means particles move in circular paths around a source.

Therefore, they neither expand nor contract away from the source